Economics 1011B
Section 4

Spring 2023

Today's Outline

- Review: Neoclassical Growth Model
  - Household and Firm Problems and FOCs
  - Competitive Equilibrium
  - Steady State
  - Transition Dynamics
- Big Push
  - Production Function
  - Phase Diagram
  - Stability of Equilibria

Today

- I want to spend most of today discussing the neoclassical growth model from last week: it's conceptually the hardest material we've seen so far in the course.
- I will also discuss the 'Big Push' model from lecture 7 - introduces some new ideas worth discussing (multiple equilibria, equilibrium stability).
- First midterm is next week in class! There will be a review session this weekend, held on Zoom and recorded: more information soon via email.

Neoclassical Growth Model: Household Problem

- Household: chooses consumption c_t and savings a_{t+1} each period to maximize lifetime utility (subject to budget constraints).

- The household's problem can be written as (with period budget constraints):

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{subject to} \quad \begin{aligned} & c_t + a_{t+1} = (1 + r_t) a_t + w_t \\ & a_0 \text{ given} \end{aligned}$$

  where the household takes initial savings a_0, the interest rate r_t, and the wage w_t as given.

- Solution to household's problem takes the form of Euler equation:

$$u'(c_t) = \beta(1 + r_{t+1}) u'(c_{t+1})$$

Neoclassical Growth Model: Firm Problem

- Firm chooses capital k_t and labor ℓ_t to maximize profits each period. They borrow at interest rate r_t to purchase capital, and they pay a wage rate w_t for each unit of labor. They sell off undepreciated capital at the end of each period.

- The firm's problem can be written:

$$\max_{k_t, \ell_t} F(k_t, \ell_t) + (1 - \delta) k_t - (1 + r_t) k_t - w_t \ell_t$$

- First-order conditions (FOCs):

$$\begin{aligned} k_t : \qquad & F_k(k_t, \ell_t) = r_t + \delta \\ \ell_t : \qquad & F_\ell(k_t, \ell_t) = w_t \end{aligned}$$

Neoclassical Growth Model: Competitive Equilibrium

- The competitive equilibrium of our economy consists of an allocation $\{c_t, a_t, k_t, \ell_t\}_{t=0}^{\infty}$ and prices $\{w_t, r_t\}_{t=0}^{\infty}$ such that:
    1. Household optimality: $\{c_t, a_t\}_{t=0}^{\infty}$ solves the household's problem, taking prices as given.
    2. Firm optimality: $\{k_t, \ell_t\}_{t=0}^{\infty}$ solve the firm's problem, taking prices as given.
    3. Financial market clearing: $a_t = k_t$ for all t (pins down r_t: supply $=$ demand for savings)
    4. Labor market clearing: $\ell_t = 1$ for all t (pins down w_t: supply $=$ demand for labor)
    5. Resource constraint holds: $k_{t+1} = (1-\delta)k_t + F(k_t, \ell_t) - c_t$ for all t
- Market clearing conditions say that supply $=$ demand. Intuitively, these conditions pin down prices: r_t is set such that $a_t = k_t$, and w_t is such that $l_t = 1$.
- Implicitly, these conditions also imply the goods market clears: output $F(K_t, \ell_t)$ goes to consumption c_t and investment (can you see how?)

Neoclassical Growth Model: Competitive Equilibrium

- Competitive equilibrium in this neoclassical growth model is characterized by:

$$
\begin{aligned}
u'(c_t) &= \beta(1 + r_{t+1})u'(c_{t+1}) && \text{Household Euler equation} \quad (1)\\
F_k(k_t, \ell_t) &= r_t + \delta && \text{Firm FOC for } k \quad (2)\\
F_\ell(k_t, \ell_t) &= w_t && \text{Firm FOC for } w \quad (3)\\
\ell_t &= 1 && \text{Labor market clearing} \quad (4)\\
a_t &= k_t && \text{Financial market clearing} \quad (5)\\
k_{t+1} &= (1 - \delta)k_t + F(k_t, \ell_t) - c_t && \text{Resource constraint} \quad (6)
\end{aligned}
$$

- Together with an initial condition k_0, these equations will tell us how the model changes over time (for all t).

- But we can combine the information in these six equations to distill the whole model into a system of two equations involving c and k.

Neoclassical Growth Model: Solving for Steady State

- Goal: derive difference equations for c and k expressing their $t+1$ values in terms of t values and constants. In particular, we want to substitute out prices (e.g. r and w) and other endogenous variables (e.g. ℓ).

- Start with Euler equation:

$$
\begin{aligned}
u'(c_t) &= \beta(1+r_{t+1})u'(c_{t+1}) && \text{(Euler equation)} \\
&= \beta(1+F_{k,t+1}(k_{t+1},\ell_{t+1})-\delta)u'(c_{t+1}) && \text{(Firm FOC for } k\text{)} \\
&= \beta(1+F_{k,t+1}(k_{t+1},1)-\delta)u'(c_{t+1}) && \text{(Labor market clearing)}
\end{aligned}
$$

- This gives an equation that implicitly characterizes c_{t+1} as a function of c_t, k_{t+1}, and constants.

Neoclassical Growth Model: Solving for Steady State

- Recall the resource constraint (law of motion for capital):

$$\begin{aligned} k_{t+1} &= (1-\delta)k_t + F(k_t, \ell_t) - c_t && \text{(Resource constraint)} \\ &= (1-\delta)k_t + F(k_t, 1) - c_t && \text{(Labor market clearing)} \end{aligned}$$

- That was easy! Given this period's capital and consumption, this difference equation tells us what capital will be next period.

- Just to recap, we have the following two difference equations:

$$u'(c_t) = \beta(1 + F_k(k_{t+1}, 1) - \delta)u'(c_{t+1}) \tag{7}$$

$$k_{t+1} = (1-\delta)k_t + F(k_t, 1) - c_t \tag{8}$$

- These play the same role that the difference equation for k played in the Solow model.

Neoclassical Growth Model: Key Equations

- To solve for the steady state, impose $c_t = c_{t+1} = c^*$ and $k_t = k_{t+1} = k^*$. The equations from the previous slide become:

$$u'(c^*) = \beta(1 + F_k(k^*, 1) - \delta)u'(c^*) \implies F_k(k^*, 1) = \beta^{-1} - 1 + \delta \tag{9}$$

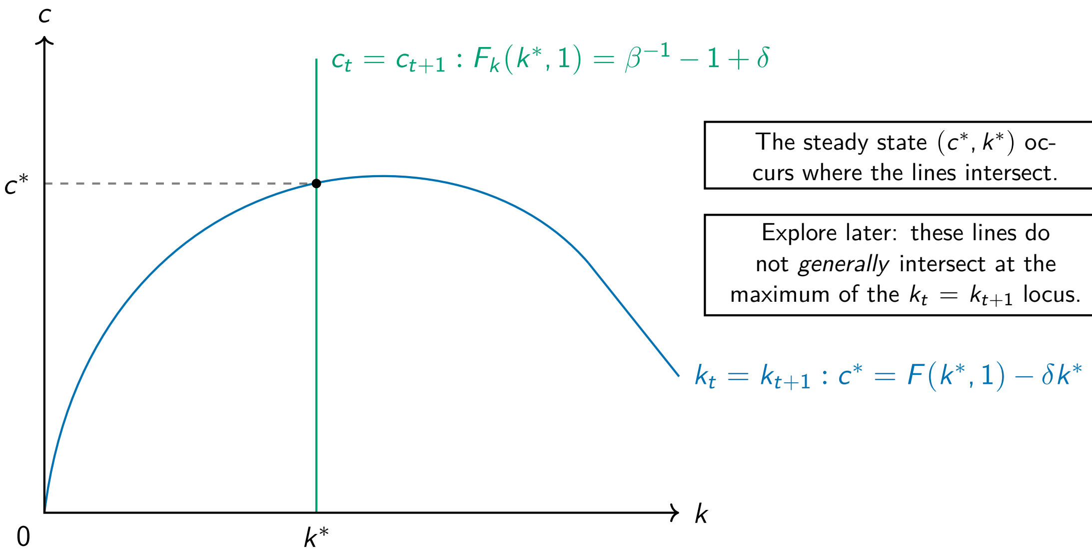
$$k^* = (1 - \delta)k^* + F(k^*, 1) - c^* \implies c^* = F(k^*, 1) - \delta k^* \tag{10}$$

- Notice that (3) alone is sufficient to solve for k^*: If the production function is Cobb-Douglas, i.e. $F(K, L) = K^\alpha L^{1-\alpha}$, can use algebra to solve for k^* in terms of parameters using only (3).

- Meanwhile, equation (4) has c^* as a function of k^*.

- If we plot these equations in (c, k) space (c as a function of k), the steady state of the neoclassical growth model is where the two lines intersect.

Neoclassical Growth Model: Phase Diagram (Steady State)



The steady state (c^*, k^*) occurs where the lines intersect.

Explore later: these lines do not *generally* intersect at the maximum of the $k_t = k_{t+1}$ locus.

Neoclassical Growth Model: Phase Diagram (Steady State)

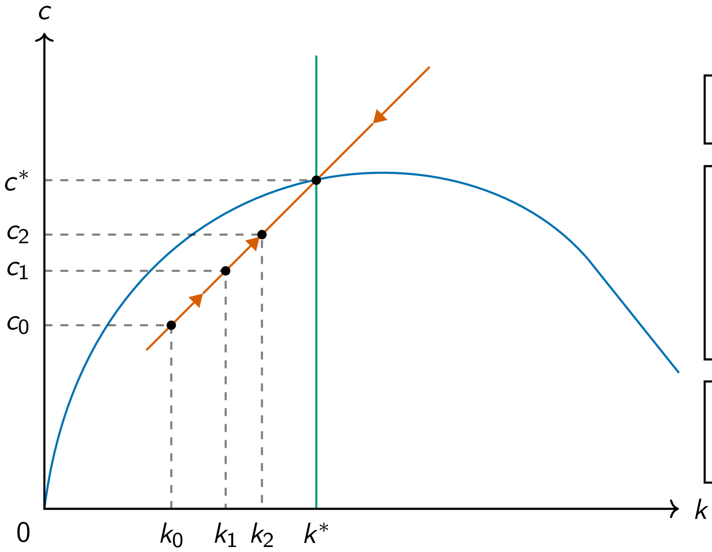
- The previous graph demonstrates how we can analyze how the neoclassical growth model's steady state depends on the rest of the model.

- We can already answer questions like, "How does a change in some parameter (e.g. $\beta \downarrow$) impact steady state consumption (c^*) or capital (k^*)?"

- What about how other endogenous variables are impacted at the steady state (e.g. r^*, w^*)? Once we've figured out (k^*, c^*), can determine the steady state value corresponding to other endogenous variables through any of our equilibrium conditions.

- Example: Suppose we want to know what happens to the steady state wage, w^*, when $\beta \downarrow$. You could figure out how such a change impacts the wage at steady state by using the other equilibrium conditions, e.g. $F_L(k_t, 1) = w_t$ (firm FOC plus labor market clearing), so $w* = F_L(k*, 1)$.

Neoclassical Growth Model: Phase Diagram (Transition Dynamics)



Consider an initial capital level $k_0 < k^*$ as drawn.

The c_0 corresponding to k_0 must lie on the red line: otherwise, consumption and capital would not converge to the steady state, but would instead shoot off (following the traffic signals) toward 0 or ∞.

After period 0, c_t and k_t follow their difference equations (staying on the red line) back to the steady state.

Neoclassical Growth Model: Dynamics

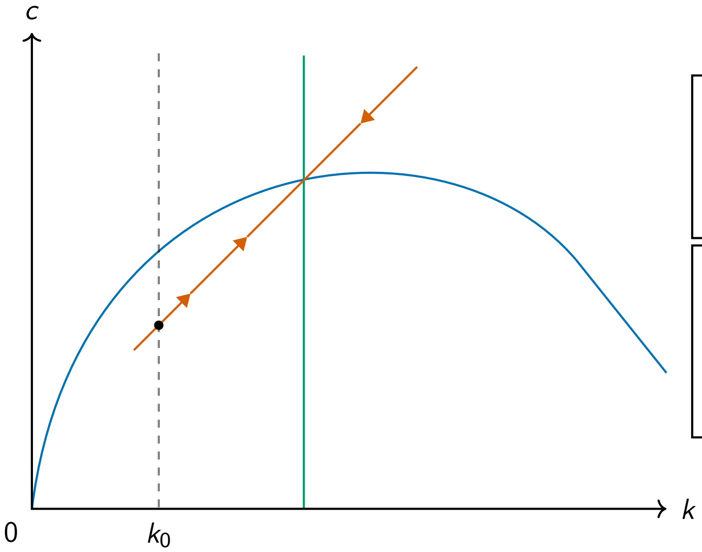
- That's it for steady states. But what happens to c, k, etc. in the short-run, i.e. as the economy converges toward steady state?

- This is a harder question, and we need to draw another line on our phase diagram.

- This line goes by many names: Ludwig called it the optimal consumption rule/policy; it also goes by the term 'policy function' (much more general term) or the 'saddle path' of this model.

- In the interest of time, in this week's section we'll presume such a line exists, and not show why it takes the shape it does - see the Section 3 slides or textbook (Kurlat/Romer) for more information on that.

Neoclassical Growth Model: Phase Diagram (Transition Dynamics)



Last week's section slides went into detail on why this red line exists. It will always have this shape (southeast to northeast quadrants of regions defined by these two lines).

Analytically, it turns out the red line does not generally have a closed form solution. But even just drawing a graph like this will help us understand how the model behaves in the transition to a new steady state.

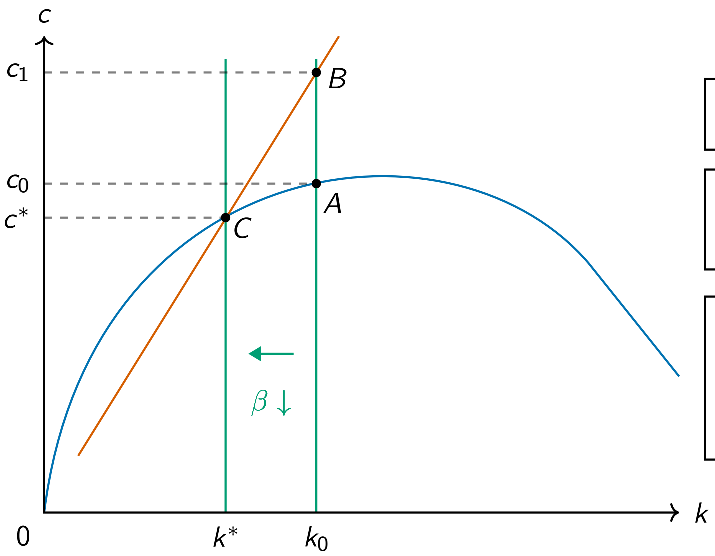
Neoclassical Growth Model: Dynamics

- Model exhibits what is sometimes called saddle path stability: for any given k_0, there is precisely one value of consumption c_0 that returns to the steady state. Any other value (even a little bit higher or lower!) would 'explode off' toward 0 capital or 0 consumption - inconsistent with utility maximization.

- As you can see: analyzing steady state dynamics (how does the steady state change if a parameter changes?) is much easier than analyzing transition dynamics.

- To answer what happens to variables in the steady state, we can use algebra (with our key difference equations, imposing steady state), or we can use the graphs and figure out how a parameter changing shifts our lines. The steady state always occurs at the intersection of the blue and green lines, but only values on the red line are attained in equilibrium.

Neoclassical Growth Model: Phase Diagram (Example)



Starting at a steady state (c_0, k_0), what happens if $\beta \downarrow$?

$\beta \downarrow$ shifts the green locus inwards (why?). The new steady state has lower c and k.

On the transition path, c is higher! (less savings). Shoot up to point B. Then follow new optimal consumpton path down to new steady state $(c*, k^*)$.

Neoclassical Growth Model: Dynamics

- How do we determine how variables like r_t and w_t adjust in response to $\beta \downarrow$ on the previous slide? The issue is they're not in the diagram directly.

- The conditions described in our definition of competitive equilibrium must hold. So we use our graph to figure out what happens to k and c when an exogenous variable changes (e.g. $\beta \downarrow$), and then use e.g. firm/household optimality conditions and market clearing conditions to deduce what's happened to the other variables.

- Example: firm FOC for k is $F_K(k_t, \ell_t) = r_t + \delta$. Since $\ell_t = 1$ when the labor market clears, this becomes $F_K(k_t, 1) = r_t + \delta$. So in equilibrium, r_t is a function of k_t: once I know how k_t changes (e.g. due to $\beta \downarrow$), I can use this relationship to see how r_t changes.

- In our example in the previous slide, r_t does not adjust immediately (when the economy moves from point A to B), but slowly rises as the economy moves down the red line from B to C (as capital shrinks toward the new, lower steady state, MPK rises, so through the firm FOC r_t must rise as well).

Neoclassical Growth Model: Recap

- We have nice, closed-form solutions for the steady state values c^* and k^* in this model. We can obtain them from setting $c_t = c_{t+1} = c^*$ and $k_t = k_{t+1} = k^*$ in our key difference equations and using algebra to rearrange.

- The red line in our phase diagram indicates the path of values that consumption and capital can take in this model. **We do not actually have a closed form equation characterizing the red line**. Solving for this (sometimes called a policy function) is beyond on the scope of this course, requiring some numerical approximation tools. So for transition dynamics in this model (assessing what happens to c, k, w, r etc if some parameter changes), the graphical approach is best. We have waved our hands a bit on this out of necessity.

- The neoclassical growth model is therefore a little delicate in the transition. You are not expected to know about saddle path stability - Ludwig intentionally did not raise these points in lecture. I'm presenting them just to provide additional context.

Big Push Model

- Now, change gears entirely and discuss the 'big push' model – essentially a very slight extension of the Solow model that Ludwig discussed in lecture.

- We will use this big push model to introduce the concept of multiple equilibria and equilibrium stability.

- So far, the models we've seen (Solow, neoclassical growth) have had a very special property, a unique steady state (given parameters) that does not depend on initial conditions.

- No matter what your initial k_0 is, you converge to some k^* that is unique given the parameters of the model in Solow and neoclassical growth.

- But in the big push model, this isn't true!

Big Push Model

- Big Push model is exactly the same ass Solow, but with a modified production function:

$$Y_t = F(K_t, L_t) = \begin{cases} K_t^{\alpha}(A_L L_t)^{1-\alpha} & \text{if } K/L < \bar{k} \\ K_t^{\alpha}(A_H L_t)^{1-\alpha} - \phi & \text{if } K/L \geq \bar{k} \end{cases}$$

  with $A_H > A_L$, $\phi > 0$, and $\bar{k}$ an exogenous 'threshold'.

- Intuition: firms with low levels of capital $k < \bar{k}$ use an inefficient technology given by the first row, firms with high levels of capital use a more efficient technology.

- This production function looks like two different Cobb-Douglas production functions pasted together! Importantly, this production function is not concave everywhere (and in fact isn't differentiable at $\bar{k}$)!

- The Big Push model still has the same key phase diagram that we had in Solow, but now it crosses the 45 degree line multiple times!

Big Push Model

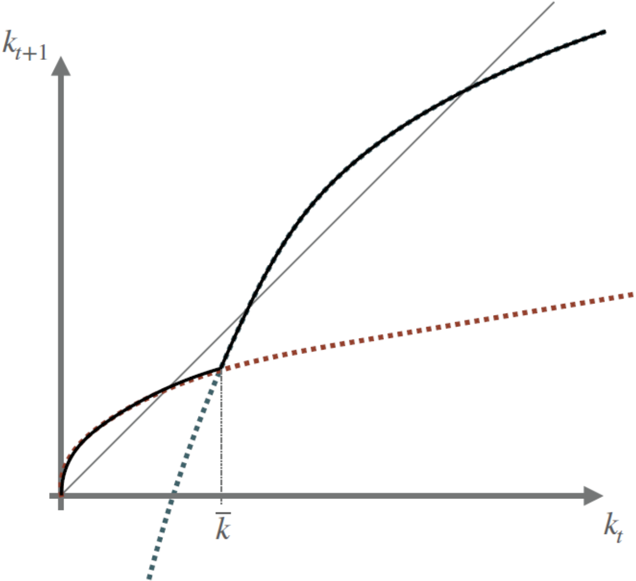
- Remainder of the model is exactly like Solow:

$$\begin{aligned}
K_{t+1} &= (1-\delta)K_t + I_t && \text{(Law of motion for capital)} \\
I_t &= s \cdot Y_t && \text{(Investment is some fraction of output)} \\
L_{t+1} &= (1+n)L_t && \text{(Law of motion for population)} \\
k_t &\equiv \tfrac{K_t}{L_t} && \text{(Definition of capital per worker)} \\
y_t &\equiv \tfrac{Y_t}{L_t} && \text{(Definition of output per worker)}
\end{aligned}$$

- Leads to the same key difference equation (with a different production function, which is now piecewise per the previous slide):

$$k_{t+1} = \frac{1}{1+n}\Big[s \cdot F(k_t, 1) + (1-\delta)k_t\Big]$$

Big Push: Phase Diagram



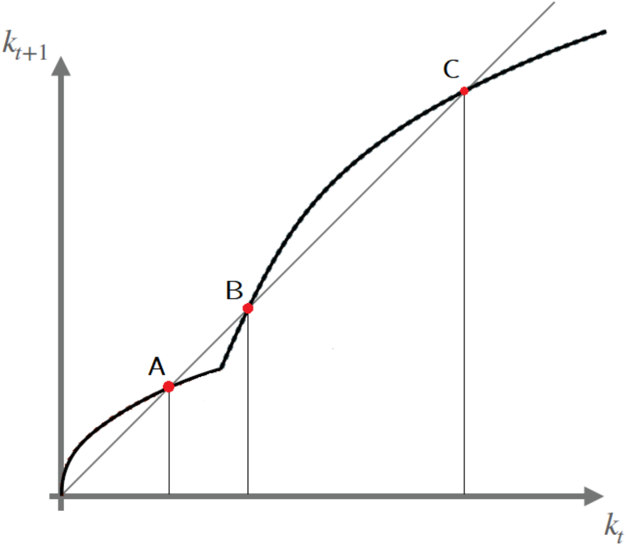
Big Push Model: Equilibrium Stability

- Key difference equation $k_{t+1} = s \cdot F(k_t, 1) + (1 - \delta)k_t$ crosses the 45 degree line $k_{t+1} = k_t$ not just once, but three times!

- Now, it's reasonable to ask which equilibrium we return to starting from any initial k_0. An equilibrium level of capital k^* is said to be stable in this model if any point in some small neighborhood $(k^* - \epsilon, k^* + \epsilon)$ will return to that equilibrium.

- There are two stable equilibria and one unstable equilibrium in this model. Can you see graphically which are which? To answer, you just need to know whether capital is growing or shrinking at each leve of k.

Big Push: Phase Diagram



Big Push Model: Equilibrium Stability

- Easiest to analyze stability case-wise, by considering how k would change on either side of all (3, in this case) equilibria.

- Notice that the production function lies above the 45 degree line below A and below the 45 degree line above A. That implies that the 'low' equilibrium, A, is stable.

- Likewise, exactly the same for the 'high' equilibrium C, which is also stable. Capital just above or below C will return to C.

- The third equilibrium, B, is unstable. The only way you can reach equilibrium B in this smodel is if you begin exactly at B! Capital just above or below B will return to the other two (stable) equilibria, A or C.

Big Push Model: Conclusion

- Pragmatically, we discussed the Big Push model because it's the simplest way we can introduce the idea of multiple equilibria (which we hadn't seen before) - an interesting property of some models. Existence and uniqueness of a steady state are both nontrivial, and sometimes our models can have multiple steady states!

- Most models we see in this course will be deliberately rigged up so as to not have multiple equilibria (original Solow, neoclassical growth); but in the Big Push model, the existence of multiple equilibria is a feature, and not a bug.

- Intuition: possible for economies to get stuck at the 'low' or 'bad' equilibrium. Small capital investment not sufficient to lift the economy from low equilibrium A to high equilibrium C: need a 'big push' that lifts the economy past the unstable equilibrium B.