# Economics 1011B
# Section 2

Spring 2023

# Today's Outline

- Mathematical Tools (Part I)
  - Taylor Approximations
  - Constrained Optimization via Lagrange Multipliers
- Consumption-Savings Model
  - Setup and Ingredients
  - The Euler Equation
  - Model Predictions
- Mathematical Tools (Part II)
  - Comparative Statics and Implicit Differentiation
- Bonus Slides
  - Infinite Horizon Consumption-Savings
  - Alternative Derivation of Euler Equation
  - Maximization Principles in Economics

# Taylor Expansions

- Taylor expansions (or Taylor approximations) will come up occasionally in 1011B.

- Taylor expansions locally approximate any differentiable function $f$ around any point $c$ using a simpler function, a polynomial of degree $n$ (we can choose $n$ and $c$).

- A first-order Taylor expansion of $f$ around a point $c$ is defined as:

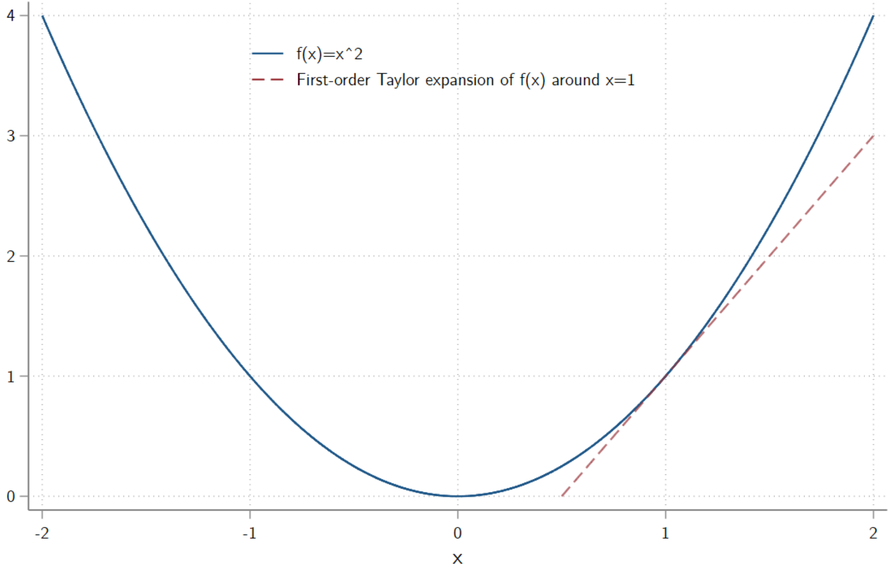
$$t(x) = f(c) + f'(c)(x - c) \tag{1}$$

- Example: We want to approximate $f(x) = x^2$ around the point $c = 1$ using a first-order Taylor expansion. We compute $f'(x) = 2x$, $f(1) = 1$, and $f'(1) = 2$. Then:

$$t(x) = 1 + 2(x - 1)$$

- Key property: for small changes in $x$ around $c$, the Taylor approximation $t(x)$ behaves like $f(x)$ (differentiable functions are 'locally linear').

# Taylor Expansions: Example



# Taylor Expansions

- We will only see first-order Taylor approximations: good enough for us.
- But if your function $f$ is differentiable at least $n$ times, you can make the approximation 'better' (albeit more complicated) by using up to an $n^{th}$-order approximation.
- The $n^{th}$ order Taylor expansion of $f(x)$ around a point $c$ is:

$$t_n(x) = \sum_{i=0}^{n} \frac{f^{(n)}(c)}{n!}(x-c)^n \tag{2}$$

  where $f^{(n)}(c)$ denotes the $n$-th derivative of $f$ evaluated at the point $c$, and the $0^{th}$ derivative is just $f$ itself.
- There are a number of other interesting properties of Taylor expansions (e.g. approximation error bounds), and we don't need to know any of them.

# Constrained Optimization

- Constrained optimization: maximizing or minimizing a function, where choice variables must respect some constraints (1011B: only work with equality constraints).

- Constrained maximization problem (single equality constraint and $n$ choice variables):

$$\max_{x_1,\ldots,x_n} f(x_1, \ldots, x_n) \quad \text{subject to} \quad g(x_1, \ldots, x_n) = 0$$

we call $f$ the *objective function*, $g$ the *constraint*, and $x_i$ the *choice variables*.

- Leading example: consumer utility maximization problem. Consumer maximizes utility function $u(\cdot)$ by choosing consumption of two goods, subject to budget constraint.

$$\max_{c_1,c_2} u(c_1, c_2) \quad \text{subject to} \quad p_1 c_1 + p_2 c_2 = y$$

where $p_1, p_2, y$ are fixed.

# Lagrange Multipliers: Motivation

- Two ways to solve constrained optimization problem like (3):
  1. Substitute constraint(s) into obj. fn.
  2. Use the Lagrange multiplier method

- Both approaches yield the same answer. I prefer (2), because it feels simpler to me. In addition, the Lagrange multiplier itself (often denoted $\lambda$) has a nice interpretation.

- The Lagrange multiplier $\lambda$ answers the question, "If I relax the constraint by one unit, how much does the value of the objective function change?"

- For instance, if a consumer is maximizing utility subject to a budget constraint, $\lambda$ tells us how much utility would increase if we gave that person 1 dollar.

# Lagrange Multipliers: Setup

- Suppose we have some function $f(x_1, ..., x_N)$. We would like to maximize (or minimize) the value of $f$ subject to some (equality) constraint $g(x_1, ..., x_N) = 0$ for a constant $c$:

$$\max_{x_1,\ldots,x_N} f(x_1, ..., x_N) \quad \text{subject to} \quad g(x_1, ..., x_N) = 0$$

- We construct a function (called the Lagrangian), the objective function plus a new parameter $\lambda$ times the constraint:

$$\mathcal{L}(x_1, ..., x_N, \lambda) = f(x_1, ..., x_N) + \lambda g(x_1, ..., x_N)$$

- Next, we calculate the derivative of $\mathcal{L}$ with respect to each of the $N + 1$ arguments $x_1, ..., x_N, \lambda$, set them equal to zero, and solve for $x_1, ..., x_N$.

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0, \ \frac{\partial \mathcal{L}}{\partial x_2} = 0, \ \ldots, \ \frac{\partial \mathcal{L}}{\partial x_N} = 0, \ \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

# Lagrange Multipliers: A Simple Example

- I am a consumer who derives utility from two goods, coffee ($c$) and donuts ($d$), which cost $p_c$ and $p_d$, respectively. I have some amount of money $y$ to spend on these goods. I take the prices $p_c$ and $p_d$ as well as my income $y$ as given.

- My problem is to choose $c$ and $d$ to maximize utility $u(c, d)$ subject to a budget constraint that says my spending must be equal to what I earn.

- We can write down this static, two-good utility maximization problem as:

$$\max_{c,d} \; u(c, d) \quad \text{subject to} \quad p_c c + p_d d = y$$

- The Lagrangian corresponding to this problem is:

$$\mathcal{L}(c, d, \lambda) = u(c, d) + \lambda \Big[ y - p_c c - p_d d \Big]$$

# Lagrange Multipliers: Simple Example (continued)

$$\max_{c,d} \; u(c,d) \quad \text{subject to} \quad p_c c + p_d d = y$$

$$\mathcal{L}(c,d,\lambda) = u(c,d) + \lambda\left[y - p_c c - p_d d\right]$$

- The first-order conditions for this Lagrangian are:

$$\frac{\partial \mathcal{L}}{\partial c} = 0 \qquad \rightarrow \qquad \partial u/\partial c - \lambda p_c = 0$$
$$\frac{\partial \mathcal{L}}{\partial d} = 0 \qquad \rightarrow \qquad \partial u/\partial d - \lambda p_d = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \qquad \rightarrow \qquad p_c c + p_d d = y$$

- These are three equations in three unknowns $(c, d, \lambda)$, and we can solve for each. Notice that the last FOC is literally just the budget constraint!

# Lagrange Multipliers: Simple Example (even more continued)

- Combining the first two first-order conditions to get rid of $\lambda$ yields:

$$\frac{\partial u/\partial c}{p_c} = \frac{\partial u/\partial d}{p_d}$$

- What does this say? The utility-maximizing choice of $c$ and $d$ is such that our marginal utility per dollar spent will be equated across the two goods.

- We don't usually need to interpret the Lagrangian multiplier itself $(\lambda)$ – but here, when the consumer is behaving optimally, $\lambda = (\partial u/\partial c)/p_c = \partial u/\partial d/p_d$.

- If we specified exactly what form $u$ has, then we could solve for $c$ and $d$ (not just their ratios) by combining either of these equations above with the budget constraint.

- For instance, if $u(c, d) = ln(c) + ln(d)$, the optimality condition is $c/d = p_d/p_c$. Combine with the budget constraint (given $p_c, p_d, y$) to solve for both $c$ and $d$.

# Consumption and Savings

- Now, turn to a model of consumption and savings.

- Goal: write down a model that expresses the economic trade-offs involved between buying stuff today vs. saving to buy stuff in the future.

- Setup: a household maximizes lifetime utility by choosing how much to consume today and tomorrow (income is exogenous). The household is allowed to freely save or borrow in the first period at a fixed interest rate $r$.

- This is not the only possible model of consumption! It is always worth asking ourselves whether this model's predictions align with the real world: indeed, a big virtue of writing down these mathematical models is that they generate testable hypotheses we can take to data.

- But this model is a good place to start: many alternative consumption models build off this one, adding e.g. borrowing constraints, behavioral agents, etc.

# Consumption and Savings: Motivating a Simple Example

- We can illustrate virtually all of the intuition behind our consumption-savings model in a simple example with only two periods, $t = 1$ and $t = 2$.

- I want to pay particular emphasis on the ingredients in the "household's problem"; in particular, the relationship between period/flow budget constraints and the lifetime budget constraint.

- It turns out that this representation will generalize extremely easy to any number $T$ periods - or even an infinite number of periods! The math becomes a little more complicated, but all of the intuition remains exactly the same.

# Consumption and Savings: Two-Period Example (In Words)

- The household lives for two periods, $t = 1, 2$, and receives income in each period, $Y_1$ and $Y_2$. For now, assume the household knows future income with certainty at the beginning of period 1 (important for us to relax this assumption, as in lecture 4!)

- The household gets utility from consumption, $u(C_t)$, in each period. In period 1 (when making savings decisions) they discount utility realized in period 2 by a factor $\beta \in [0, 1]$. Assume utility function satisfies $u' > 0$ and $u'' < 0$ (diminishing marginal utility).

- The household chooses how much to consume in each of the two periods, given their income / budget constraints. The household can freely save or borrow in the first period at a fixed interest rate $r$.

- Normalize the price of the consumption good to 1 - putting the budget constraint in real terms. This is without loss of generality: income and savings are measured in units of the consumption good.

# Consumption and Savings: Two-Period Example (In Math)

- One way to write down the household's problem is using 'flow' budget constraints:

$$\max_{C_1, C_2} \; u(C_1) + \beta u(C_2) \quad \text{subject to} \quad \begin{aligned} C_1 + S &= Y_1 \\ C_2 &= Y_2 + (1+r)S \end{aligned}$$

- Combining the two flow budget constraints to eliminate $S$:

$$\max_{C_1, C_2} \; u(C_1) + \beta u(C_2) \quad \text{subject to} \quad C_1 + \tfrac{1}{1+r} C_2 = Y_1 + \tfrac{1}{1+r} Y_2$$

- These problems are equivalent. We could even solve them both with Lagrangians: with two constraints, you need two multipliers $(\lambda_1, \lambda_2)$. But we'll work through with lifetime budget constraint for now.

# Consumption and Savings: Solving Two-Period Example

- The Lagrangian corresponding to this problem is:

$$\mathcal{L}(C_1, C_2, \lambda) = u(C_1) + \beta u(C_2) + \lambda\Big[Y_1 + \tfrac{1}{1+r}Y_2 - C_1 - \tfrac{1}{1+r}C_2\Big]$$

- The first-order conditions (FOCs) for this problem are:

$$\frac{\partial \mathcal{L}}{\partial C_1} = 0 \quad \rightarrow \quad u'(C_1) = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial C_2} = 0 \quad \rightarrow \quad \beta u'(C_2) = \tfrac{1}{1+r}\lambda$$

- Combining the equations above to eliminate $\lambda$, we derive what's called the Euler equation:

$$u'(C_1) = \beta(1+r)u'(C_2) \qquad \text{(Euler equation)}$$

- In order to figure out exactly what $C_1$ and $C_2$ are, we need functional form assumptions on the utility function (and we also need to combine this with the budget constraint - two equations, two unknowns).

# Consumption-Savings: The Euler Equation

- The Euler equation, which captures the trade-off between consuming today vs. tomorrow for an optimizing consumer, is a really important take-away of this model. We'll see it come up in more complicated models.

- It is derived by combining the FOCs for consumption in periods $t$ and $t+1$ (this works no matter how many periods there are!).

- The Euler equation characterizes how consumption is split between two periods (e.g. $C_2/C_1$). Equivalently, the Euler equation pins down the growth rate of consumption between periods $t$ and $t+1$.

- To solve for $C_2$ or $C_1$ in terms of model parameters, we need another equation - the budget constraint.

# Consumption-Savings: Predictions

- Believe it or not, our two-period consumption savings model has a lot of strong, testable predictions that we could take to the real world.

- Prediction #1: Timing of income doesn't matter, conditional on the present discounted value of income. Example: suppose $r = 0$, so the PDV of income is just $Y_1 + Y_2$. All that matters is $Y_1 + Y_2$ - not how income is divided across periods.

- Prediction #2: Consumption today responds immediately if income in the future changes. Moreover, consumption responds much more to movements in permanent income than temporary income (see Lecture 4). The household is 'forward-looking'. We will work through this example shortly.

- Prediction #3: Consumption is super sensitive to changes in the interest rate (Euler).

- Prediction #4: Consumption smoothing / excess smoothness of consumption: so important, the whole next slide is on it!

# Consumption-Savings: Consumption Smoothing

- Consumption smoothing: households are "consumption-smoothers" in the sense that consumption growth does not depend on income.

- Consider special case: $\beta = 1$, $r = 0$.
  - Euler equation $u'(C_1) = u'(C_2)$ implies $C_1 = C_2$ since $u'' < 0$ (why? draw graph of $u'(c)$)
  - Importantly, household will choose $C_1 = C_2$ no matter no matter what $Y_1$, $Y_2$ are!
  - "Perfect" consumption smoothing more generally whenever $\beta(1 + r) = 1$!

- Why is it optimal for household to perfectly smooth consumption in this case? Diminishing marginal utility ($u'' < 0$) crucial: household earns more lifetime utility by smoothing consumption rather than going ham in one period

- Discounting $\beta < 1$ and nonzero interest rates $r > 0$ only accentuates consumption smoothing. Smaller $\beta$: more impatient, consume earlier. Higher $r$: tilt consumption toward the second period (savings is more valuable and borrowing more costly).

# Consumption and Savings: Quick Note on Utility

- So far, we have not specified an exact functional form for utility $u(C_t)$, except that it is increasing and concave ($u' > 0$, $u'' < 0$). In other words, we're only assuming diminishing marginal utility.

- Often, we do not actually need to specify a form for utility to derive economically useful results: we will work through one such example in a few slides.

- But often it is handy to specify a concrete example just to work through numerical examples.

# Consumption and Savings: Intuition

- For instance, suppose $u(C_t) = ln(C_t)$. Then $u'(C_t) = 1/C_t$. Plugging this into our Euler equation, we get:

$$\frac{1}{C_1} = \beta(1+r)\frac{1}{C_2} \rightarrow \frac{C_2}{C_1} = \beta(1+r)$$

- Note that this equation is specific to the 'log utility' function. If we have a different functional form for utility, we get a different equation.

- Let's interpret this equation together (with log utility) for a couple cases.

- What does $\beta(1+r) < 1$ mean? What does it imply regarding how this individual allocates consumption across periods? What about $\beta(1+r) > 1$?

- What about $\beta(1+r) = 1$? How does this relate to the idea of 'consumption smoothing'?

# Comparative Statics and Implicit Differentiation

- Suppose you want to compute a comparative static: you want to find out how an endogenous variable in your model changes as an exogenous variable or parameter changes.

- For instance, in our two-period consumption-savings model, how much does consumption $C_1$ change when period 2 income $Y_2$ changes? In math, we are interested in computing $\partial C_1 / \partial Y_2$.

- To do this, you will need to do implicit differentiation. If you haven't seen it before, it requires no additional knowledge beyond the usual single-variable differential calculus tricks, like the power rule, the chain rule, and the product rule.

- Implicit differentiation is best illustrated by example (next slide).

# Comparative Statics and Implicit Differentiation

- Let's consider what happens to $C_1$ when $Y_2$ increases ($\partial C_1 / \partial Y_2$).

- Start with the Euler equation, rearranged just a bit so zero is on one side:

$$u'(C_1) - \beta(1+r)u'(C_2) = 0$$

- The idea of implicit differentiation is to note that the endogenous variables ($C_1$ and $C_2$) are "implicit functions" of $Y_2$ (how? $C_1$ and $C_2$ are optimized s.t. budget constraint).

- Differentiate the Euler equation with respect to $Y_2$:

$$\frac{\partial}{\partial Y_2}\left[u'(C_1) - \beta(1+r)u'(C_2)\right] = 0 \qquad \text{(set up derivative)}$$

$$u''(C_1)\frac{\partial C_1}{\partial Y_2} - \beta(1+r)u''(C_2)\frac{\partial C_2}{\partial Y_2} = 0 \qquad \text{(evaluate derivative, use chain rule)}$$

# Comparative Statics and Implicit Differentiation

- We want to solve for $\frac{\partial C_1}{\partial Y_2}$, but how can we get rid of $\frac{\partial C_2}{\partial Y_2}$? Can rearrange lifetime budget constraint to express $C_2$ as a function of $C_1$:

$$C_1 + \frac{1}{1+r} C_2 = Y_1 + \frac{1}{1+r} Y_2 \rightarrow C_2 = (1+r) Y_1 + Y_2 - (1+r) C_1$$

- Differentiating this (budget constraint) with respect to $Y_2$ yields:

$$\frac{\partial C_2}{\partial Y_2} = 1 - (1+r) \frac{\partial C_1}{\partial Y_2}$$

- Combining this with the equation at the top of this slide and then solving for $\frac{\partial C_1}{\partial Y_2}$:

$$\frac{\partial C_1}{\partial Y_2} = \frac{\beta(1+r) u''(C_2)}{u''(C_1) + \beta(1+r)^2 u''(C_2)}$$

- We need to take a stand on $u$ to fully solve this, but it is positive if $u'' < 0$ (why?).
  $\implies$ $C_1$ rises when $Y_2$ goes up in our model (for any well-behaved utility function!).

# Consumption-Savings: Generalizing to $T$ periods and beyond

- It isn't very difficult to generalize this model to $T > 2$ periods, or even an infinite number of periods (infinite horizon).

- We don't have the time to go through it now: I have a few slides detailing the math (for one way to tackle this problem) in the bonus slides. I did it in a different way than Ludwig did, but we get the same answer.

- Next week, we'll use our consumption-savings model as a buiding block for neoclassical growth. Intuition is exactly the same.

# Consumption-Savings: Wrapping Up

- We've developed a nice model for thinking about how people might choose to consume today versus consume tomorrow.

- There are a lot of assumptions that go into it, and for the purposes of studying macroeconomic events (like recessions), it seems incomplete - income is totally exogenous!

- Next week: embed our consumption-savings model inside a richer model where firms produce/sell goods to consumers. First full-blown macro model: the neoclassical growth model.

# Bonus Slides

# Bonus Slides: Infinite Horizon Consumption-Savings

- I want to revisit some of the content from Lecture 4, and in particular show you how I think about breaking down an infinite horizon ($T \to \infty$) consumption-savings model.
- I want to emphasize: the economics remains the same! We get an identical Euler equation, and we even use the same tools of constrained optimization.
- As Ludwig mentioned, we can solve this model with either period budget constraints or a lifetime budget constraint; and for each of those choices, we can solve by substituting the constraints in the objective function or using a Lagrangian.
- I'll show you how you can solve the model with period budget constraints and a Lagrangian - just to show that it's not that hard!

# Bonus Slides: Infinite Horizon Consumption-Savings

- Infinite-horizon consumption-savings problem: time is discrete, indexed by $t = 0, 1, 2, \ldots$.

- Individual starts with initial wealth $S_0$ and receives income $Y_t$ each period. Must choose consumption $C_t$ each period (which implicitly pins down savings, $S_{t+1}$).

- The period $t$ (for every $t = 0, 1, 2\ldots$) budget constraint is:

$$C_t + S_{t+1} = (1 + r)S_t + Y_t$$

- Note: there are an infinite number of these! Following Ludwig's slides, we can combine with $t + 1$ etc budget constraints to yield the lifetime budget constraint:

$$\sum_{t=0}^{\infty} \frac{C_t}{(1+r)^t} = (1 + r)S_0 + \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t}$$

# Bonus Slides: Infinite Horizon Consumption-Savings

- Maximization problem, written with period budget constraints (notice, infinite number!):

$$\max_{\{C_t, S_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t) \quad \text{s.t.} \quad C_t + S_{t+1} = (1+r)S_t + Y_t \quad \forall\, t = 0, 1, 2, \ldots$$

- Looks a little cumbersome: but it's the same problem we had before. The real distinction is that there are an infinite number of choice variables, and we're making savings decisions every period.

- Very technically, in the infinite horizon model, we need an extra condition - often called the 'no Ponzi condition' - to make sure that the household will eventually pay back all debt they have. We'll gloss over this for now because it is not essential to derive the Euler equation. In the two-period model, this was easy: we just assumed all borrowing / savings were paid back in the final period - but there is no final period here.

# Bonus Slides: Infinite Horizon Consumption-Savings

- Lagrangians can accomodate multiple equality constraints - we just need a separate Lagrangian multiplier $\lambda$ for each constraint.

- Fortunately for us, the period budget constraints all look the same except for the date $t$, so it's easy to write:

$$\mathcal{L}(C_0, C_1, ..., \lambda_1, \lambda_2, ...) = \sum_{t=0}^{\infty} \beta^t u(C_t) + \sum_{t=0}^{\infty} \lambda_t \Big[(1+r)S_t + Y_t - C_t - S_{t+1}\Big]$$

- Note that as before, I like writing my constraints in Lagrangians as 'income minus spending' - that ensures the sign of the multiplier $\lambda_t$ corresponds to the question, "How much does a dollar of extra income in period $t$ increase lifetime utility?"

- That's a lot of constraints and choice variables! An infinite number, in fact. We only need to take two: with respect to $C_t$ and $S_{t+1}$. We'll let $t$ be arbitrary!

# Bonus Slides: Infinite Horizon Consumption-Savings

- First-order conditions for $C_t$ and $S_{t+1}$:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \rightarrow \quad \beta^t u'(C_t) = \lambda_t$$
$$\frac{\partial \mathcal{L}}{\partial S_{t+1}} = 0 \rightarrow \quad \lambda_t = \lambda_{t+1}(1+r) = 0$$

- Draw attention to the FOC for $S_{t+1}$ above - deriving this is nontrivial and trips up students when they first see it all the time (I remember being confused by this way back when, too!).

- The key thing here is that $S_{t+1}$ appears not just in the period $t$ budget constraint, but also in the period $t+1$ constraint! Hence, we have $\lambda_{t+1}$. Make sure you see this: it's because $S$ appears twice with different dates in the constraint.

# Bonus Slides: Infinite Horizon Consumption-Savings

- Need to eliminate $\lambda_t$ and $\lambda_{t+1}$. Note that since we solved the FOC of $\mathcal{L}$ with respect to any arbitrary $C_t$, we also know the FOC for $C_{t+1}$.

- In words, the FOC for $C_t$ was $\beta^t u'(C_t) = \lambda_t$. Thus, $\beta^{t+1} u'(C_{t+1}) = \lambda_{t+1}$. Substituting in to the FOC for $S_{t+1}$, we can eliminate both $\lambda$'s:

$$\beta^t u'(C_t) = \beta^{t+1} u'(C_{t+1})(1 + r)$$

- Divide both sides by $\beta^t$ and we're done - we've derived the Euler equation in the infinite-horizon model!

- Recall the Euler equation tells us something about consumption growth between periods $t$ and $t + 1$. Since we let $t$ be arbitrary, we're done! We don't need to solve for all $C$'s, since we could solve for an arbitrary pair.

# Bonus Slides: Infinite Horizon Consumption-Savings

- What do we gain from the infinite-horizon model? Well, first and most importantly, we will need it next week when we discuss neoclassical growth.

- In a growth model (just like in Solow), we need an infinite horizon to characterize how growth changes over time. Growth is inherently dynamic, and two periods is just insufficient! So we'll need this infinite-horizon framework next week.

- Other than that, the intuition is all (obviously, from the Euler equation) more or less the same. It is easier to see in this model why the permanent income hypothesis is embodied by this model. Suppose $\beta(1 + r) = 1$; then just as in our two-period model, consumption will be constant over time by the Euler equation.

- That immediately means that if I give you a dollar today, you will spread it out over your entire lifetime... and when $T$ is really large or infinite, that implies you will consume next to nothing. That seems odd – some of us, like your TF, spent all of their COVID stimulus immediately, which does not seem to accord with this model!

# Consumption-Savings: Perturbation to Derive Euler Equation

- Useful, very general alternative to derive and interpret Euler equation: 'perturbation approach'. You don't need to learn this! Just here as an alternate perspective.

- Suppose you have a consumption bundle $(C_1, C_2)$ that satisfies the lifetime budget constraint $Y_1 + \frac{Y_2}{1+r} = C_1 + \frac{C_2}{1+r}$. Question: is $(C_1, C_2)$ optimal, in the sense that it maximizes lifetime utility s.t. budget constraint?

- Consider 'perturbation': reduce $C_1$ by a tiny amount $\epsilon > 0$, increase $C_2$ by $\epsilon(1+r)$. Utility decreases by $u'(C_1) \times \epsilon$ in $t = 1$, increases by $\beta u'(C_2) \times \epsilon(1+r)$ at $t = 2$.

- Change in utility $= \Big[ -u'(C_1) \times \epsilon \Big] + \Big[ \beta u'(C_2) \times \epsilon(1+r) \Big]$
  - If positive: $(C_1, C_2)$ cannot be optimal, since perturbation does better.
  - If negative: $(C_1, C_2)$ cannot be optimal, since opposite perturbation does better.

- So original bundle $(C_1, C_2)$ can only be optimal when the change in utility is zero. Set change in utility $= 0$, cancel $\epsilon$ to yield Euler equation: $u'(C_1) = \beta(1+r)u'(C_2)$.

# Bonus Slides: Defense of Maximization Principles

- Why do we assume that agents maximize utility or profit in economics?

- Paul Samuelson was an old economist (and Harvard PhD) who, more than anyone else, was responsible for the amount and style of math you see in this course (so blame him).

- Samuelson's defense of 'maximization principles' in his Nobel lecture (1970, link):

> *... the plumb-line trajectory of a falling apple and the elliptical orbit of a wandering planet may be capable of being described by the optimizing solution for a specifiable programming problem. But no one will be tempted to ... attribute to the apple or the planet freedom of choice and consciously deliberative minimizing. Nonetheless, to say Galileos ball rolls down the inclined plane as if to minimize the integral of action, or to minimize Hamiltons integral, does prove to be useful to the observing physicists, eager to formulate predictable uniformities of nature.*

# Bonus Slides: Defense of Maximization Principles

- Samuelson's idea - which he hammered down in a textbook that ended up influencing generations of economists - is that the strength of economic models is that they allow us to generate testable hypotheses about the world. Assuming our agents are maximizing something allows us to generate sharp predictions that can be taken to data.

- Clearly, our models do not accurately describe the data-generating process for the real economy (way too complicated!). But perhaps they will allow us to think clearly about specific aspects of a problem that we are interested in, like why the government spending by a dollar might cause economic activity to increase by more than (or less than) a dollar.

- Good economic models often yield surprising insights about the ways that agents in our model economies interact that are not obvious when we lay out the problem. Models force us to be explicit about our assumptions and trace out their logical conclusions.